1 Four different objects can be arranged in $4! = 24$ different ways.

2 A teacher must occupy the first position. There are 3 choices for this position. There are five more people to be arranged in $5!$ ways. Therefore, using the multiplication principle there are a total of

$$3 \times 5! = 360$$

arrangements.

3 a There are five digits to choose from, and each can be used as many times as you like. Therefore, using the multiplication principle, there are

$$5 \times 5 \times 5 = 125$$

possibilities.

b There are 5 choices for the first digit, 4 for the second and 3 for the third. Therefore, using the multiplication principle, there are

$$5 \times 4 \times 3 = 60$$

possibilities.

4 a $1 + 2 \times 4 = 9$.

b

$$\begin{aligned} 1 + (1 \times 3 \times 2 \times 2) + (3 \times 2 \times 2 \times 1) &= 1 + 12 + 12 \\ &= 25 \end{aligned}$$

5 a $4! = 4 \times 3 \times 2 \times 1 = 24$

b

$$\begin{aligned} \frac{6!}{4!} &= \frac{6 \times 5 \times 4!}{4!} \\ &= 6 \times 5 \\ &= 30 \end{aligned}$$

c

$$\begin{aligned} \frac{8!}{6!2!} &= \frac{8 \times 7 \times 6!}{6! \times 2!} \\ &= \frac{8 \times 7}{2} \\ &= 28 \end{aligned}$$

d

$$\begin{aligned} {}^{10}C_2 &= \frac{10!}{8!2!} \\ &= \frac{10 \times 9 \times 8!}{8! \times 2!} \\ &= \frac{10 \times 9}{2} \\ &= 45 \end{aligned}$$

6 a There are fives choices for the first position, four for the second, three for the third and two for the fourth. This gives a total of

$$5 \times 4 \times 3 \times 2 = 120$$

arrangements.

b Five children can be arranged in five spaces in $5! = 120$ ways.

7 a There are a total of 5 items and these can be arranged in $5! = 120$ different ways.

b We group the three mathematics books together so that we now have just three items: $\{M_1, M_2, M_3\}, P_1, P_2$. These three items can be arranged in $3! = 6$ different ways. However, the three mathematics books can be arranged within the group in $3! = 6$ different ways. This gives a total of $6 \times 6 = 36$ different arrangements.

8 a Although the question states that there are no restrictions, we still can't have the zero in the first position or else the number wouldn't be a five-digit number. Therefore there are on 4 possibilities for the first digit. The remaining four digits can be arranged in $4!$ ways. This gives a total of $4 \times 4! = 96$ numbers.

b If the number is divisible by 10 then the last digit must be zero. The remaining four digits can be arranged without restriction in $4! = 24$ different ways.

c If the number is greater than 20000 then the first digit can one of three options: $2, 3$ or 4. The remaining four digits can be arranged in $4!$ ways. This gives a total of $3 \times 4! = 72$ different numbers.

d Obviously the last digit must be either $0, 2$ or 4. We need to consider two cases. **Case 1:** If the last digit is 0 then the remaining four digits can be arranged without further restriction in $4! = 24$ ways. **Case 2:** If the last digit is 2 or 4 then there are two possibilities for the final digit. As the first digit cannot be 0 there remains just 3 possibilities. The remaining three digits can be arranged in $3! = 6$ different ways. This gives a total of $3 \times 3! \times 2 = 36$ numbers.
Using the addition principle, there are a total of $24 + 36 = 60$ different numbers.

9 There are five items in total, of which a group of three are alike and a group of two are alike. These can be arranged in

$$\frac{5!}{2! \times 3!} = 10$$

different ways.

10a Three children from six can be selected in ${}^6C_3$ ways. This gives,

$$\begin{aligned} {}^6C_3 &= \frac{6!}{3!3!} \\ &= \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} \\ &= \frac{6 \times 5 \times 4}{6} \\ &= 20. \end{aligned}$$

b Two letters from twenty-six can be selected in ${}^{26}C_2$ ways. This gives,

$$\begin{aligned} {}^{26}C_2 &= \frac{26!}{2!24!} \\ &= \frac{26 \times 25 \times 24!}{2! \times 24!} \\ &= \frac{26 \times 25}{2} \\ &= 325. \end{aligned}$$

c Four numbers out of ten can be selected in ${}^{10}C_4$ ways. This gives,

$$\begin{aligned} {}^{10}C_4 &= \frac{10!}{4!6!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6!}{4! \times 6!} \\ &= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \\ &= 210. \end{aligned}$$

d Three sides out of eight can be selected in ${}^8C_3$ ways. This gives,

$$\begin{aligned} {}^8C_3 &= \frac{8!}{3!5!} \\ &= \frac{8 \times 7 \times 6 \times 5!}{3! \times 5!} \\ &= \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \\ &= 56. \end{aligned}$$

11a Two elements from eight can be chosen in ${}^8C_2$ ways. This gives,

$$\begin{aligned} {}^8C_2 &= \frac{8!}{2!6!} \\ &= \frac{8\times 7\times 6!}{2!\times 6!} \\ &= \frac{8\times 7}{2\times 1} \\ &= 28. \end{aligned}$$

b Each set must contain the number 8. We are still to choose two more numbers from the set $\{1, 2, \ldots, 7\}$. These can be chosen in ${}^7C_2$ ways. This gives,

$$\begin{aligned} {}^7C_2 &= \frac{7!}{2!5!} \\ &= \frac{7\times 6\times 5!}{2!\times 5!} \\ &= \frac{7\times 6}{2\times 1} \\ &= 21. \end{aligned}$$

c A set with eight elements will have $2^8 = 256$ subsets (including the empty set, and the entire set).

12 Three boys can be selected from five in ${}^5C_3$ ways. Two girls can be selected from four in ${}^4C_2$ ways. Using the multiplication principle, we can make both selections in

$$ {}^5C_3 \times {}^4C_2 = 60$$

ways.

13 There are three cases to consider.

Labor	Liberal	Selections
1	3	${}^4C_1 \times {}^5C_3$
2	2	${}^4C_2 \times {}^5C_2$
3	1	${}^4C_3 \times {}^5C_1$

This gives a total of 120 selections.

14 Label three holes with the colours blue, green and red.

$$\boxed{\text{B}}\ \boxed{\text{G}}\ \boxed{\text{R}}$$

Clearly, selecting six balls is not sufficient as you might pick two balls of each colour. Now select seven balls and place each sock in the hole whose label corresponds to the colour of the sock. As there are seven balls and three holes, the Pigeonhole Principle guarantees that some hole contains at least three balls. Therefore the answer is seven.

15 Label fifty boxes with the numbers

$$\boxed{1 \text{ or } 99}\ \boxed{2 \text{ or } 98}\ \cdots\ \boxed{49 \text{ or } 51}\ \boxed{50}$$

Selecting 50 different numbers is not sufficient as you might pick one number belonging to each box. Now select 51 numbers, and place each one in its corresponding hole. As there are 51 numbers and 50 holes, some hole contains 2 numbers. The two numbers in this hole are different, and so their sum is 100.

16 Let A and B be the sets comprising of multiples 2 and 3 respectively. Clearly $A \cap B$ consists of the multiples of 2 and 3, that is, multiples of 6. Therefore, $|A| = 60$, $|B| = 40$ and $|A \cap B| = 20$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 60 + 40 - 20 \\ &= 80. \end{aligned}$$

17 If n is odd, then $n = 2k+1$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n^2 + n &= (2k+1)^2 + (2k+1) \\ &= 4k^2 + 4k + 1 + 2k + 1 \\ &= 4k^2 + 6k + 2 \\ &= 2(2k^2 + 3k + 1) \end{aligned}$$

is even.

18 Since m and n are consecutive integers, we know that $n - m = 1$. Therefore,

$$\begin{aligned} n^2 - m^2 &= (n-m)(n+m) \\ &= 1 \times (n+m) \\ &= n + m. \end{aligned}$$

19a Converse: If n is odd, then $5n+3$ is even.

b If n is odd, then $n = 2k+1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 5n + 3 &= 5(2k+1) + 3 \\ &= 10k + 5 + 3 \\ &= 10k + 8 \\ &= 2(5k + 4) \end{aligned}$$

is even.

c Contrapositive: If n is even, then $5n+3$ is odd.

d If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 5n + 3 &= 5(2k) + 3 \\ &= 10k + 3 \\ &= 10k + 2 + 1 \\ &= 2(5k + 1) + 1 \end{aligned}$$

is odd.

20 **Method 1:** Suppose that $x+1$ were rational. Then there would be $p, q \in \mathbb{Z}$ such that

$$x + 1 = \frac{p}{q}.$$

It follows that,

$$\begin{aligned} x &= \frac{p}{q} - 1 \\ &= \frac{p}{q} - \frac{q}{q} \\ &= \frac{p-q}{q}. \end{aligned}$$

Since $p - q \in \mathbb{Z}$ and $q \in \mathbb{Z}$ this implies that x is rational. This is a contradiction.
Method 2: Suppose that $x+1$ were rational. Then

$$x = \overbrace{(x+1)}^{\text{rational}} - \overbrace{1.}^{\text{rational}}$$

Therefore, x is the difference of two rational numbers, which is rational. This is a contradiction.

21 Suppose on the contrary that 6 can be written as the difference of two perfect squares m and n.Then

$$\begin{aligned} 6 &= n^2 - m^2 \\ 6 &= (n-m)(n+m) \end{aligned}$$

The only factors of 5 are $1, 2, 3$ and 6. And since $n+m > n-m$ we need only consider two cases.
Case 1: If $n-m=2$ and $n+m=3$ then we add these two equations together to give $2n=5$. This means that $n=5/2$, which is not a whole number.
Case 2: If $n-m=1$ and $n+m=6$ then we add these two equations together to give $2n=7$. This means that $n=7/2$, which is not a whole number.

22 ($\Rightarrow$) Suppose n is odd. Then $n=2k+1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n+1 &= 3(2k+1)+1 \\ &= 6k+3+1 \\ &= 6k+4 \\ &= 2(3k+2) \end{aligned}$$

is even.
($\Leftarrow$) We will prove the equivalent contrapositive statement.
Contrapositive: If n is even, then $3n+1$ is odd.
Proof. Suppose n is even. Then $n=2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n+1 &= 3(2k)+1 \\ &= 6k+1 \\ &= 2(3k)+1 \end{aligned}$$

is odd.

23a This is false, for each of 2 and 5 are prime numbers and so too is $2+5=7$.

b Any number $x \le 1$ will provide a counter-example. For example, let $x=1/2$. Then,

$$x^3 = 1/8 < 1/4 = x^2.$$

Alternatively, you could let $x=-1$. Then,

$$x^3 = -1 < 1 = x^2.$$

24 We need to show that the opposite is true. That is, for all $n \in \mathbb{N}$, the number $25n^2-9$ is a composite number. To see this, note that
$25n^2-9 = (5n-3)(5n+2)$
And since $5n-3 \ge 2$ and $5n+2 > 2$, we have expressed $25n^2-9$ as the product of two natural numbers greater than 1.

25a $\boxed{P(n)}$
$2+4+\cdots+2n = n(n+1)$
$\boxed{P(1)}$
If $n=1$ then

$$\text{LHS} = 2$$

and

$$\text{RHS} = 1 \times 2 = 2.$$

Therefore $P(1)$ is true.
$\boxed{P(k)}$
Assume that $P(k)$ is true so that

$$2+4+\cdots+2k=k(k+1). \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$
$=2+4+\cdots+2k+2(k+1)$
$=k(k+1)+2(k+1) \quad (\text{by } (1))$
$=(k+1)(k+2)$
$=(k+1)((k+1)+1)$
Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

11^n-6 is divisible by 5

$\boxed{P(1)}$

If $n=1$ then $11^1-6=5$ is divisible by 5. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$11^n-6=5m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}11^{k+1}-6 &= 11\times 11^k-6\\ &= 11\times(5m+6)-6 \quad (\text{by } (1))\\ &= 55m+66-6\\ &= 55m+60\\ &= 5(11m+12)\end{aligned}$$

is divisible by 5. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

26a 90° (angle subtended by a diameter at the circumference)

b 54° ($\angle BAC=90^\circ$, (angle subtended by a diameter at the circumference)
$x=(180-36-90)^\circ=54^\circ$ (angle sum of triangle)

c $x=80^\circ$ (angle subtended at the circumference is half the angle subtended by the same arc at the centre)

d $x=220^\circ$ (angle subtended at the circumference is half the angle subtended by the same arc at the centre)

e $x=96^\circ$ (opposite angles in a cyclic quadrilateral),
$y=70^\circ$ (opposite angles in a cyclic quadrilateral)

f 46° (alternate segment theorem)

27a $\angle DAB=(180-80)^\circ=100^\circ$
$\angle ABD=40^\circ$ (isosceles triangle)

b $\angle YDA=100^\circ$ (alternate angles)
$\angle BDY=(100+40)^\circ=140^\circ$

c $\angle ABC=(40+90)^\circ=130^\circ$
$\therefore \angle BCD=50^\circ$ (co-interior angles)

28a $\angle CBA=90^\circ$, (angle subtended by a diameter at the circumference)
$\angle CAB=(180-90-52)^\circ=38^\circ$ (angle sum of triangle)

b $\angle BDA=52^\circ$ (angles subtended by the same arc)
$\angle PAD=52^\circ$ (alternate angles, $PQ \parallel DB$)

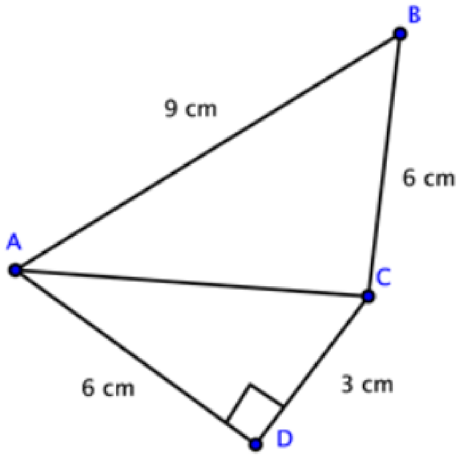
c $\angle ABD = 22^\circ$ (alternate angles, $PQ \parallel DB$)
$\therefore \angle CBD = (90 - 22)^\circ = 68^\circ$

29a $PA \times PC = PB \times PD$ (power of a point)
$6x = 24$
$x = 4$

b $CA \times CB = CP \times CD$ (power of a point)
$2x^2 = 45$
$x = \sqrt{\frac{45}{2}}$
$= \frac{3\sqrt{10}}{2}$

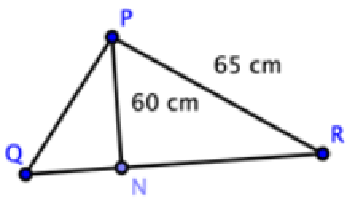
c $MJ \times MK = ML \times MN$ (power of a point)
$2x = 24$
$x = 12$

30



$AC^2 = 36 + 9 = 45$ (Pythagoras' theorem)
Consider,
$AC^2 + BC^2 = 45 + 36 = 81$
$\therefore AC^2 + BC^2 = AB^2$
$\therefore \angle ACB = 90^\circ$ (converse of Pythagoras' theorem)

31



a $NR^2 = 65^2 - 60^2 = 625 \therefore NR = 25$ $\triangle QNP \sim \triangle PNR$ (AAA)

$\therefore \frac{QP}{PR} = \frac{NP}{NR}$

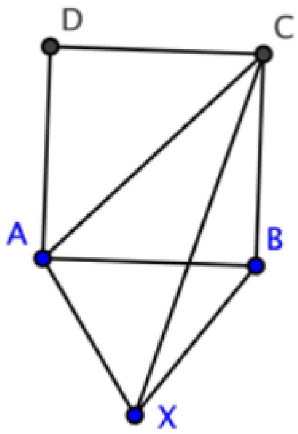
$\therefore QP = \frac{60}{25} \times 65$

$\therefore QP = 156$

b $QN^2 = 156^2 - 60^2$
$\therefore QN = 144$

c From part a $\mathrm{NR} = 25$

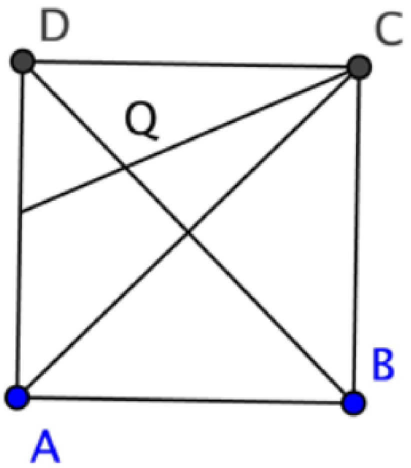
32



$\angle XAC = (45 + 60)^\circ = 105$ (Equilateral triangle and diagonal of a square bisect angle)
$\triangle XBC$ is isosceles and $\angle XBC = (60 + 90)^\circ = 150^\circ$
$\therefore$ in $\triangle XBC, \angle CXB = \angle XCB = 15^\circ \therefore \angle ACX = (90 - 15 - 45)^\circ = 30^\circ$

33 Angle sum of n-sided polygon $= (n - 2)180^\circ$
For 11 sides : $(11 - 2) \times 180^\circ = 9 \times 180^\circ$
For 5 sides: $(5 - 2) \times 180^\circ = 3 \times 180^\circ$

34



$\angle DCQ = \angle QCA = 22.5^\circ$
$\angle QCB = (45 + 22.5^\circ = 67.5^\circ$
$\angle CQB = (45 + 22.5^\circ = 67.5^\circ$ (exterior angle of a triangle) $\therefore \triangle BQC$ is isosceles
$\therefore BQ = CB = CD$

35 The right angled-triangle with shorter sides 28 cm and 45 cm has hypotenuse length 53 cm. Therefore the given triangle is an obtuse-angled triangle.